A nonsingular solution of the edge dislocation in the gauge theory of dislocations

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Abstract
A (linear) nonsingular solution for the edge dislocation in the translational gauge theory of defects is presented. The stress function method is used and a modified stress function is obtained. All field quantities are globally defined and the solution agrees with the classical solution for the edge dislocation in the far field. The components of the stress, strain, distortion and displacement fields are also defined in the dislocation core region and they have no singularity there. The dislocation density, moment and couple stress for an edge dislocation are calculated. The solutions for the stress and strain fields obtained here are in agreement with those obtained by Gutkin and Aifantis through an analysis of the edge dislocation in the strain gradient elasticity. Additionally, the relation between the gauge theory and Eringen's so-called nonlocal theory of dislocations is given.

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1. Introduction

The gauge theory of dislocations [1–22] is a very attractive field of research. The theory of dislocations is, in these theories, considered as a translational gauge theory ($T(3)$-gauge theory). An important goal is to find modified solutions of the stress and strain fields of screw and edge dislocations in this framework because the 'classical' elasticity of dislocations contains singularities at the dislocation line. Therefore, the classical solutions cannot be applied within the dislocation core region. This is unfortunate since the core region is the most important from the point of dislocation and failure mechanics. The difference in these gauge models is the form of the $T(3)$-gauge Lagrangian. Edelen *et al* [6–8, 10] used the simplest form quadratic in the $T(3)$-gauge field strength. In [1–4, 13] the gauge Lagrangian was proposed in the three-dimensional Hilbert–Einstein form. The most general dislocation gauge Lagrangian for an isotropic and anisotropic material was proposed in [1, 2].

A correct gauge theoretical solution of the stress field of a screw dislocation is given in [2–4, 9, 10]. This solution has no singularity at the dislocation line and the far-field stress is that of a classical screw dislocation.

Up to now, the situation is less satisfactory for the edge dislocation. In the Edelen model [6–8], the far field of the stress σ_{zz} is different from the standard one. In fact, the far-field stress σ_{zz} is equal to $\nu^{-1}(\sigma_{xx} + \sigma_{yy})$ instead of $\nu(\sigma_{xx} + \sigma_{yy})$ (see also [4]). Unfortunately, no explicit gauge solution for the edge dislocation was given in [6–8]. Moreover, the structure of the gauge Lagrangian used by Edelen *et al* requires, in principle, an asymmetric force stress [2]. Malyshev [4] used the Hilbert–Einstein Lagrangian as a gauge Lagrangian and found the stress field of a straight edge dislocation with a modified asymptotic behaviour which differs from the classical edge dislocation. In fact, the stress field has no singularity at the dislocation line and has an oscillatory behaviour with slowly decreasing amplitude far from the dislocation core. His solution can be considered as a complicated gauge defect with modified asymptotics instead of an elementary defect [4].

On the other hand, there are other non-standard continuum models of defects, e.g., the Peierls–Nabarro model [23–26], the nonlocal continuum model [27–31] and the strain gradient elasticity [32–38]. The force stresses calculated in these theories are symmetric even in the core. In the case of a screw dislocation, the solution of the force stresses in nonlocal elasticity [28, 30, 31] and gradient elasticity [37, 38] agrees with the gauge theoretical one [2–4, 9, 10]. Additionally, the strain fields calculated in gradient theory [35, 37, 38] and gauge theory [2, 3, 9, 10] coincide. In the case of an edge dislocation, the solution in nonlocal elasticity was given in an integral form by means of a special two-dimensional nonlocal kernel [30]. Using another two-dimensional nonlocal kernel Eringen [27] gave a different solution of the straight edge dislocation. The dislocation stress field in gradient elasticity [36–38] was obtained in a closed analytical form by using the Fourier transform method; it is equal to zero at the dislocation line and beyond the dislocation core, the classical and the gradient solution coincide. In a quasi-continuum model [31, 39–42], a modified stress field of an edge dislocation and a screw dislocation was also obtained. In this approach the stress field has no divergence at the dislocation line and at large distances it contains a decreasing oscillatory contribution. Dislocations have also been considered in a Cosserat media (see, e.g., [43–45]). The force stresses of dislocations calculated within the Cosserat theory are, in general, asymmetric. The solutions of edge and screw dislocations [46–52] found for the elastic strain and stress fields differ from the classical solutions but still posses singularities at the dislocation line.

A common feature of gauge theory, nonlocal elasticity, strain gradient theory and theory of Cosserat media of defects is that a characteristic length scale enters the constitutive laws. In the gauge theory of defects and in the theory of Cosserat and multipolar media, this length is a material property which carries with it all of the difference between solutions with or without moment stresses. Its influence might become important as dimensions of the body or wavelengths diminish to the order of the characteristic inner length. By means of this characteristic length scale, it is possible to define the dislocation core radius in a straightforward manner [2].

The question arises: can we find a gauge theoretical solution for the edge dislocation which has the correct far-field stress and is related to any solution of another framework? The aim of this paper is to find an answer to this question, which means to construct a correct gauge theoretical solution of a straight edge dislocation. Additionally, the relation to the nonlocal theory and strain gradient elasticity of an edge dislocation will be given. Here we restrict our considerations to the case of linear dislocation theory.

2. The translational gauge invariant formulation of dislocation theory

For the undeformed reference state of the body, the indices $i = 1, 2, 3$ are used. The final coordinates of the deformed state are labelled by the indices $a = 1, 2, 3$. In presence of defects, the deformed state of the material is characterized by nonholonomic coordinates. In linear theory we may use $\delta_a^{\ i}$, δ_{ij} and δ_{ab} to raise and lower the indices.

In elasticity theory the state of an elastic body is not changed under a rigid translation and rotation (global gauge transformations). In elasticity the (compatible) distortion is given as a proper gradient

$$\beta_i^a = \partial_i u^a. \tag{2.1}$$

The distortion tensor is dimensionless. The basic idea of gauge theory of defects is now that the distortion has to be invariant not only under the global gauge transformation but also if the gauge group is applied locally. In this way, one obtains the incompatible distortion in the framework of translational gauge theory of dislocations as follows:

$$\beta_i^a = \partial_i u^a + \phi_i^a. \tag{2.2}$$

It is locally translational invariant (gauge invariant) due to

$$u^a \longrightarrow u^a + \tau^a(x) \qquad \phi_i^a \longrightarrow \phi_i^a - \partial_i \tau^a(x) \tag{2.3}$$

where $\tau^a(x)$ are local translations [2]. Here u^a is the displacement field from the undeformed to the deformed configuration and ϕ_i^a is the proper incompatible part of the distortion. The incompatible distortion (2.2) can be understood as the (minimal) replacement of the compatible distortion (2.1) in the $T(3)$-gauge theory

$$\partial_i u^a \longrightarrow \partial_i u^a + \phi_i^a. \tag{2.4}$$

Then, the (incompatible) strain tensor turns out to be[1]

$$E_{ij} = \tfrac{1}{2}\big(\delta_{ai}\beta_j^a + \delta_{aj}\beta_i^a\big) \equiv \beta_{(ij)}. \tag{2.5}$$

The skew-symmetric part of the distortion defines the elastic rotation tensor

$$\omega_{ij} = \tfrac{1}{2}\big(\delta_{ai}\beta_j^a - \delta_{aj}\beta_i^a\big) \equiv \beta_{[ij]} \tag{2.6}$$

and the elastic rotation vector

$$\omega_i \equiv -\tfrac{1}{2}\epsilon_i^{\ jk}\delta_{aj}\beta_k^a. \tag{2.7}$$

The elastic strain energy of an isotropic media is defined by

$$W = \frac{1}{2}\mu\left(\delta^{ik}\delta^{jl} + \delta^{il}\delta^{jk} + \frac{2\nu}{1-2\nu}\delta^{ij}\delta^{kl}\right)E_{ij}E_{kl} \tag{2.8}$$

where μ is the shear modulus and ν is Poisson's ratio. It is now invariant under local translations. Obviously, no elastic rotation enters relation (2.8) and, therefore, the elastic rotation has no elastic response quantity. The force stress can be derived from the strain energy

$$\sigma^{ij} = \frac{\partial W}{\partial E_{ij}} = 2\mu\left(E^{ij} + \frac{\nu}{1-2\nu}\delta^{ij}E_k^{\ k}\right). \tag{2.9}$$

It is the elastic response quantity to the strain tensor. Formula (2.9) expresses the (incompatible) stress–strain relation (generalized Hooke's law).

It is well known that nontrivial traction boundary problems in the variational formulation of the gauge theory of defects can be formulated by means of a so-called null Lagrangian

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[1] We will be using the notation $A_{(ij)} \equiv \tfrac{1}{2}(A_{ij} + A_{ji})$ and $A_{[ij]} \equiv \tfrac{1}{2}(A_{ij} - A_{ji})$.

[8, 10, 53]. When the null Lagrangian is added to the Lagrangian of elasticity, it does not change the Euler–Lagrange equation (force equilibrium) because the associated Euler–Lagrange equation, $\partial_j \overset{\circ}{\sigma}_a{}^j = 0$, is identically satisfied. After minimal replacement, $\partial_i u^a \rightarrow \beta_i^a$, the null Lagrangian

$$W_{\text{bg}} = \partial_j \left(\overset{\circ}{\sigma}_a{}^j u^a \right) = \left(\partial_j \overset{\circ}{\sigma}_a{}^j \right) u^a + \overset{\circ}{\sigma}_a{}^j \partial_j u^a \longrightarrow \overset{\circ}{\sigma}_a{}^j \beta^a{}_j \tag{2.10}$$

gives rise to a background stress tensor $\overset{\circ}{\sigma}_a{}^j$ which can be considered as the nucleation field in the gauge theory of defects [10].

As usual, we use the translational field strength (torsion) to define the dislocation density tensor [1–3, 16, 17, 54–57]

$$T_{ij}^a = \partial_i \beta_j^a - \partial_j \beta_i^a \tag{2.11}$$

which has the dimension of an inverse length. The dislocation density tensor is a fundamental quantity in plasticity because the dislocation is the elementary carrier of plasticity. This field is a physical state quantity which, at least in principle, can be measured without knowing anything about the history of the body. It is possible with the help of high resolution transmission electron microscopes (HRTEM) to see single dislocations with their core configuration in crystals. Since dislocations change the energy of the body (crystal), they should appear in the total Lagrangian. The dislocation Lagrangian, which we use, contains only the translational gauge field strength and is given by

$$\mathcal{L}_{\text{disl}} = -\tfrac{1}{4} T_{ij}^a H_a{}^{ij} \tag{2.12}$$

where the moment stress $H_a{}^{ij}$ is the response quantity to dislocation density. This means that at all positions where the dislocation density is nonzero, localized moment stresses are also present. It is convenient to perform an irreducible decomposition of the torsion, from which we can construct the most general isotropic constitutive law between the dislocation density and the moment stress in the following way [1, 2]:

$$H_{aij} = \sum_{I=1}^{3} a_I \,^{(I)}T_{aij}. \tag{2.13}$$

Here a_1, a_2 and a_3 are new material coefficients. They have the dimension of a force. The irreducible decomposition of the torsion under the rotation group $SO(3)$ reads

$$T_{aij} = {}^{(1)}T_{aij} + {}^{(2)}T_{aij} + {}^{(3)}T_{aij}. \tag{2.14}$$

The tensor, the trace and the axial tensor pieces are defined by (see also [2, 58])

$$^{(1)}T_{aij} = T_{aij} - {}^{(2)}T_{aij} - {}^{(3)}T_{aij} \qquad \text{(tentor)} \tag{2.15}$$

$$^{(2)}T_{aij} := \tfrac{1}{2} \left(\delta_{ai} T^l{}_{lj} + \delta_{aj} T^l{}_{il} \right) \qquad \text{(trator)} \tag{2.16}$$

$$^{(3)}T_{aij} := \tfrac{1}{3} \left(T_{aij} + T_{ija} + T_{jai} \right) \qquad \text{(axitor)}. \tag{2.17}$$

Then, the moment stress tensor reads

$$H_{aij} = -2 \frac{\partial \mathcal{L}_{\text{disl}}}{\partial T^{aij}} = c_1 T_{aij} + c_2 (T_{ija} + T_{jai}) + c_3 \left(\delta_{ai} T^l{}_{lj} + \delta_{aj} T^l{}_{il} \right) \tag{2.18}$$

with the abbreviations

$$c_1 := \tfrac{1}{3}(2a_1 + a_3) \qquad c_2 := \tfrac{1}{3}(a_3 - a_1) \qquad c_3 := \tfrac{1}{2}(a_2 - a_1). \tag{2.19}$$

The Euler–Lagrange equations for the Lagrangian $\mathcal{L} = \mathcal{L}_{\text{disl}} - W + W_{\text{bg}}$ turn out to be [2]

$$\frac{\delta\mathcal{L}}{\delta u^a} \equiv \partial_j \sigma_a{}^j = 0 \qquad \text{(force equilibrium)} \tag{2.20}$$

$$\frac{\delta\mathcal{L}}{\delta\phi^a{}_j} \equiv \partial_i H_a{}^{ij} = \hat{\sigma}_a{}^j \qquad \text{(moment equilibrium)} \tag{2.21}$$

with the so-called effective stress tensor

$$\hat{\sigma}_a{}^j := \sigma_a{}^j - \overset{\circ}{\sigma}_a{}^j \tag{2.22}$$

which drives the dislocation fields. The field equations (2.20) and (2.21) determine the fields $\beta^a{}_j$ and $T^a{}_{ij}$ which represent the geometric (or kinematic) degrees of freedom. Note that in our approach the dislocation density is not given, *a priori*, as delta function. In this field theoretical context, we are able to calculate even the dislocation core. The static responses are the fields $\sigma_a{}^j$ and $H_a{}^{ij}$.

The LHS of (2.21) can be written in terms of the distortion tensor

$$\partial^i H_{aij} = c_1(\Delta\beta_{aj} - \partial^i\partial_j\beta_{ai}) + c_2(\partial^i\partial_j\beta_{ia} - \partial_a\partial^i\beta_{ij} + \partial_a\partial^i\beta_{ji} - \Delta\beta_{ja})$$
$$+ c_3\left(\partial_a\partial^i\beta_{ij} - \partial_a\partial_j\beta^i{}_i + \delta_{aj}\Delta\beta^i{}_i - \delta_{aj}\partial^i\partial^k\beta_{ik}\right). \tag{2.23}$$

In the general case, the gauge theory of dislocations for isotropic materials contains three new material coefficients which can be used to define three internal characteristic lengths. In order to obtain only one from the three coefficients we need a suitable choice for the moment stress tensor H_{aij} and the coefficients a_1, a_2 and a_3, respectively. By means of such a choice we want to find a gauge theoretical solution of symmetric force stresses for the edge dislocation. If we use the choice[2]

$$a_2 = \frac{1+\nu}{1-\nu}a_1 \qquad a_3 = -\frac{a_1}{2} \qquad \Longrightarrow \qquad c_1 = \frac{a_1}{2} \qquad c_2 = -\frac{a_1}{2} \qquad c_3 = \frac{\nu}{1-\nu}a_1 \tag{2.24}$$

the localized moment stress tensor reads

$$H_{aij} = \frac{a_1}{2}\left(T_{aij} - T_{ija} - T_{jai} + \frac{2\nu}{1-\nu}\left(\delta_{ai}T^l{}_{lj} + \delta_{aj}T^l{}_{il}\right)\right). \tag{2.25}$$

With the relations $H_{ak} = \frac{1}{2}\epsilon_{ijk}H_a{}^{ij}$ and $T^a{}_{ij} = \epsilon_{ijk}\alpha^{ak}$, and by using the conventional (microscopic) dislocation density tensor $\alpha^{ai} = \epsilon^{ijk}\partial_j\beta^a{}_k$ and the Nye [59] tensor[3] κ_{aj},

$$\kappa_{aj} = \alpha_{ja} - \tfrac{1}{2}\delta_{aj}\alpha^k{}_k \qquad \alpha_{aj} = \kappa_{ja} - \delta_{aj}\kappa^k{}_k \tag{2.26}$$

we obtain from (2.18) the relation

$$H_{aj} = a_1\left(\alpha_{aj} - \frac{1}{2}\delta_{aj}\alpha^k{}_k + \frac{2\nu}{1-\nu}\alpha_{[aj]}\right) = a_1\left(\kappa_{ja} + \frac{2\nu}{1-\nu}\kappa_{[ja]}\right). \tag{2.27}$$

Alternatively, the dislocation Lagrangian (2.12) can be written in the form

$$\mathcal{L}_{\text{disl}} = -\tfrac{1}{2}\alpha^{ai}H_{ai}. \tag{2.28}$$

[2] Note that Malyshev [4] used the so-called Einstein choice $a_2 = -a_1$ and $a_3 = -\frac{a_1}{2}$ for a (modified) edge dislocation. By applying the stress function method, he found a gauge theoretical solution for the edge dislocation in terms of the Bessel and Neumann functions and the modified Bessel functions. His solution of a (modified) edge dislocation has no singularity at $r = 0$ but differs asymptotically from the classical edge dislocation. In detail, the symmetric stress field, the closure failure and the dislocation density show a decaying behaviour with oscillations. Additionally, his solution violates in the dislocation core region the plane-strain condition $E_{zz} = 0$.

[3] Here the tensor κ_{aj} is a microscopic quantity.

By using (2.27) the LHS of (2.21) reads

$$\partial^i H_{aij} = a_1 \left\{ \Delta\beta_{(aj)} - \partial^i\partial_j\beta_{(ai)} + \frac{1}{1-\nu}\partial_a\partial^i\beta_{[ij]} \right.$$
$$\left. - \frac{\nu}{1-\nu}\left((\partial_a\partial_j - \delta_{aj}\Delta)\beta^i{}_i + \delta_{aj}\partial^i\partial^k\beta_{ik} - \partial_a\partial^i\beta_{(ij)}\right) \right\}. \tag{2.29}$$

Obviously, the strain and the rotation tensors appear in (2.29) because the distortion is not symmetric. It is recalled that in the so-called Hilbert–Einstein choice, only the strain tensor enters the moment equilibrium (see [2, 3]).

We now define the rotation gradient[4]

$$k^a{}_{ij} = \partial^a\beta_{[ij]}. \tag{2.30}$$

It has the dimension of reciprocal of length. The rotation gradient describes a 'lattice-curvature'. It contains two pieces. To see this we rewrite (2.30),

$$k_{ijk} = (\partial_k E_{ij} - \partial_j E_{ik}) + \tfrac{1}{2}(T_{ijk} - T_{jki} - T_{kij}). \tag{2.31}$$

The first one is caused by local (incompatible) elastic strain and the other one by dislocations (contortion) [57]. The second piece may be identified with the plastic 'lattice-curvature' which cannot be realized without dislocations. We add the following gauge-invariant Lagrangian which looks like a rotation gradient Lagrangian of a Cosserat medium (see, e.g., [62]):

$$\mathcal{L}_{\text{grad}} = -\tfrac{1}{2}k^a{}_{ij}\mu_a{}^{ij}. \tag{2.32}$$

The response quantity to the rotation gradient is a so-called 'Cosserat' couple stress for an isotropic material[5] (see, e.g., [62]):

$$\mu_{a[ij]} = -\frac{\partial\mathcal{L}_{\text{grad}}}{\partial k^{a[ij]}} = d_1 k_{a[ij]} + d_2(k_{i[ja]} + k_{j[ai]}) + d_3\left(\delta_{ai}k^l{}_{[lj]} + \delta_{aj}k^l{}_{[il]}\right). \tag{2.33}$$

This constitutive relation uses the three rotation gradients, which are invariant under the $SO(3)$, and is analogous to the invariant constitutive relation for the torsion (2.18). We find

$$\partial^i\mu_{i[aj]} = d_1\Delta\beta_{[aj]} + (d_2 - d_3)(\partial^i\partial_j\beta_{[ia]} - \partial^i\partial_a\beta_{[ij]}). \tag{2.34}$$

Another gauge-invariant Lagrangian, which we add, is given in terms of the torsion and the rotation gradient

$$\mathcal{L}_{\text{grad–disl}} = -\tfrac{1}{2}d_0 k^a{}_{ij}T_a{}^{ij}. \tag{2.35}$$

Equation (2.35) may be considered as a kind of an interaction Lagrangian between dislocations and the 'lattice-curvature'. We define the response quantity

$$\lambda_{aij} = -\frac{\partial\mathcal{L}_{\text{grad–disl}}}{\partial(\partial^i\beta^{aj})} = d_0\left(k_{aij} + \frac{1}{2}T_{iaj}\right) \tag{2.36}$$

with

$$\partial^i\lambda_{aij} = d_0\partial^i\left(\partial_a\beta_{[ij]} + \tfrac{1}{2}(\partial_a\beta_{ij} - \partial_j\beta_{ia})\right). \tag{2.37}$$

The coefficients d_0, d_1, d_2 and d_3 have the dimensions of force.

If we use the Lagrangian $\mathcal{L} = \mathcal{L}_{\text{disl}} + \mathcal{L}_{\text{grad}} + \mathcal{L}_{\text{grad–disl}} - W + W_{\text{bg}}$, the moment equilibrium reads

$$\partial^i(H_{aij} + \lambda_{aij} + \mu_{i[aj]}) = \hat{\sigma}_{aj} \tag{2.38}$$

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[4] DeWit [60, 61] called such a rotation gradient elastic bend–twist.
[5] The relations between the constants in (2.33) and those in [62] are: $d_1 = 2\alpha_2$, $d_2 = \alpha_3$, $d_3 = -\alpha_1$.

which can be decomposed into its symmetric and antisymmetric parts. For the symmetric part of the moment equilibrium we obtain the equation

$$\partial^i (H_{(aj)i} + \lambda_{(aj)i}) + \hat{\sigma}_{(aj)} = 0. \tag{2.39}$$

By using the choice of the coefficient

$$d_0 = -\frac{a_1}{(1 - \nu)} \tag{2.40}$$

formula (2.39) can be rewritten in terms of the strain tensor, only, according to

$$a_1 \left\{ \Delta E_{ij} - \partial^k \partial_{(i} E_{j)k} - \frac{\nu}{1 - \nu} \left((\partial_i \partial_j - \delta_{ij} \Delta) E^k_{k} + \delta_{ij} \partial^k \partial^l E_{kl} - \partial^k \partial_{(i} E_{j)k} \right) \right\} = \hat{\sigma}_{ij}. \tag{2.41}$$

This equation can be interpreted as the (static) field equation in the elastoplastic theory of dislocations. It is derived from the combination of several Lagrangians—equations (2.12), (2.32), (2.35)—and the consideration of the null Lagrangian (2.10) and the strain energy (2.8) in combination with the special choices of material constants (2.24) and (2.40). The antisymmetric part of the moment equilibrium[6]

$$\partial^i (H_{[aj]i} - \lambda_{[a|i|j]} - \mu_{i[aj]}) = 0 \tag{2.42}$$

can be satisfied with the help of the choice of the coefficients

$$d_1 = 0 \qquad d_2 - d_3 = \frac{d_0}{2}. \tag{2.43}$$

The physical interpretation of equation (2.42) is the following. Because we have required that the force stress should be symmetric, $\sigma_{[ij]} = 0$, we have had to introduce the couple stresses $\lambda_{[a|i|j]}$ and $\mu_{i[aj]}$ in order to fulfil the antisymmetric moment equilibrium (2.42). Consequently, (2.42) expresses the moment equilibrium between the antisymmetric moment stresses $H_{[aj]i}$, $\lambda_{[a|i|j]}$ and $\mu_{i[aj]}$. The fact that the moment stress is the specific response to the presence of dislocations and that the field equation (2.38) expresses the moment equilibrium is a hint that the gauge theory of dislocations may be connected with the theory of a constrained, anholonomic Cosserat media with symmetric force stress. The dislocations bring the anholonomity into the picture. A similar remark in this direction was given by Kluge [45]. It should also be possible to extend this theory to a more general dislocation theory in a general Cosserat continuum with antisymmetric force stress and ten material constants, which is not the aim of the present paper. However, for antisymmetric force stress and vanishing dislocation density, the moment equilibrium equation recovers the form of the Cosserat equilibrium equation under zero external moments.

Finally, we may define the hyperstress

$$M_a^{ij} \equiv -\frac{\partial \mathcal{L}}{\partial (\partial_i \beta^a_{j})} = H_a^{ij} + \lambda_a^{ij} + \mu^i_{a}^{j}. \tag{2.44}$$

This hyperstress tensor is close to Maugin's so-called 'Piola–Kirchhoff' hyperstress [63]. Such a hyperstress tensor was also used by Toupin for the theory of nonsimple elastic materials [64]. Thus, the dislocation theory given in this paper can be considered as an anholonomic higher-order gradient theory in nonsimple materials. If $a_1 = 0$, there are no effects of hyperstresses and couple stresses.

With the help of the inverse of Hooke's law

$$E_{ij} = \frac{1}{2\mu} \left(\sigma_{ij} - \frac{\nu}{1 + \nu} \delta_{ij} \sigma^k_{k} \right) \tag{2.45}$$

[6] Indices which are exempt from antisymmetrization are enclosed by vertical bars.

and using the equilibrium condition $\partial^j \sigma_{ij} = 0$, we obtain from equation (2.41) the inhomogeneous Helmholtz equation as field equation for every component of the stress tensor

$$(1 - \kappa^{-2}\Delta)\sigma_{ij} = \overset{\circ}{\sigma}_{ij} \qquad \kappa^2 = \frac{2\mu}{a_1}. \tag{2.46}$$

It is important to note that (2.46) agrees with the field equation for the stress field in nonlocal elasticity [28, 30, 31] and gradient elasticity [37, 38]. The factor κ^{-1} has the physical dimension of length and, therefore, it defines an internal characteristic length (dislocation length scale). By using (2.45) and (2.46) we obtain an inhomogeneous Helmholtz equation for the strain fields,

$$(1 - \kappa^{-2}\Delta)E_{ij} = \overset{\circ}{E}_{ij} \tag{2.47}$$

where $\overset{\circ}{E}_{ij}$ is the background strain tensor. Equation (2.47) is similar to the equation for strain of the gradient theory used by Gutkin and Aifantis [35–38], if we identify κ^{-2} with the corresponding gradient coefficient (see, e.g., equation (4) in [37]). Note that only for compatible strain, $\phi_{ij} = 0$, one can deduce an inhomogeneous Helmholtz equation for the displacement from (2.47), which is analogous to the equation for the displacement in gradient elasticity (see equation (3) in [37]).

The conditions on a gauge theoretical solution of dislocations in this framework are: (i) the stress field should have no singularity at $r = 0$, (ii) the far-field stress ought to be the classical stress field. We use condition (ii) because the classical stress fields are well established and in good agreement with physical observations, e.g., the investigation of stresses around dislocations by means of the photoelasticity method (see, e.g., [65–67]).

3. The gauge theoretical solution of the straight edge dislocation

3.1. The stress, strain and displacement fields in an infinitely extended body

Let us now derive the correct dislocation fields within the translational gauge theory. We take Cartesian coordinates so that the z-axis is along the dislocation line and the x-axis is along the Burgers vector: $b^x = b$, $b^y = b^z = 0$. The extra half-plane lies in the plane $x = 0$. In order to satisfy the equilibrium condition, we use the second-order stress function f and specialize to the plane problem of an edge dislocation by setting $\partial_z \equiv 0$. We are using the so-called stress function ansatz [26]

$$\sigma_{ij} = \begin{pmatrix} \partial^2_{yy} f & -\partial^2_{xy} f & 0 \\ -\partial^2_{xy} f & \partial^2_{xx} f & 0 \\ 0 & 0 & p \end{pmatrix} \tag{3.1}$$

with

$$p = \nu\Delta f \tag{3.2}$$

where Δ denotes the two-dimensional Laplacian $\partial^2_{xx} + \partial^2_{yy}$. Relation (3.2) comes from the requirement: $E_{zz} = 0$ (plane strain). Additionally, the strain is given in terms of the stress function as

$$E_{ij} = \frac{1}{2\mu} \begin{pmatrix} \partial^2_{yy} f - \nu\Delta f & -\partial^2_{xy} f & 0 \\ -\partial^2_{xy} f & \partial^2_{xx} f - \nu\Delta f & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{3.3}$$

We use the classical stress field of a straight edge dislocation in terms of the Airy stress function as the background stress

$$\overset{\circ}{\sigma}_{ij} = \begin{pmatrix} \partial^2_{yy}\chi & -\partial^2_{xy}\chi & 0 \\ -\partial^2_{xy}\chi & \partial^2_{xx}\chi & 0 \\ 0 & 0 & \nu\Delta\chi \end{pmatrix}. \tag{3.4}$$

The Airy stress function [68–73]

$$\chi = -Ay\ln r \qquad A = \frac{\mu b}{2\pi(1-\nu)} \tag{3.5}$$

fulfils the following inhomogeneous bipotential (or biharmonic) equation:

$$\Delta\Delta\chi = -4\pi A\partial_y\delta(r) \tag{3.6}$$

where $\delta(r)$ is the two-dimensional Dirac delta function and $r^2 = x^2 + y^2$. Substituting (3.1) and (3.4) into (2.46) we get

$$(\Delta - \kappa^2)f = \kappa^2 Ay\ln r. \tag{3.7}$$

We make the ansatz

$$f = -Ay\ln r + f_{(1)} \tag{3.8}$$

and obtain

$$(\Delta - \kappa^2)f_{(1)} = 2A\partial_y\ln r. \tag{3.9}$$

Now we set

$$f_{(1)} = 2A\partial_y f_{(2)} \tag{3.10}$$

with

$$(\Delta - \kappa^2)f_{(2)} = \ln r \tag{3.11}$$

and we get the solution

$$f_{(2)} = -\frac{1}{\kappa^2}(\ln r + K_0(\kappa r)). \tag{3.12}$$

Finally, we obtain the solution of (3.7),

$$f = -A\left(y\ln r + \frac{2}{\kappa^2}\partial_y(\ln r + K_0(\kappa r))\right). \tag{3.13}$$

Then the solution of the modified stress function of an edge dislocation with Burgers vector $b\|x$ reads[7]

$$f = -\frac{\mu b}{2\pi(1-\nu)}y\left\{\ln r + \frac{2}{\kappa^2 r^2}(1 - \kappa r K_1(\kappa r))\right\} \tag{3.14}$$

where the first piece is the Airy stress function and K_n is the modified Bessel function of the second kind and $n = 0, 1, \ldots$ denotes the order of this function.

By means of equations (3.1) and (3.14), the modified stress of a straight edge dislocation is given by

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[7] For an edge dislocation with the Burgers vector $b\|y$ the modified stress function is given by

$$f = \frac{\mu b}{2\pi(1-\nu)}x\left\{\ln r + \frac{2}{\kappa^2 r^2}(1 - \kappa r K_1(\kappa r))\right\}.$$

$$\sigma_{xx} = -\frac{\mu b}{2\pi(1-\nu)}\frac{y}{r^4}\left\{(y^2+3x^2)+\frac{4}{\kappa^2 r^2}(y^2-3x^2)-2y^2\kappa r K_1(\kappa r)-2(y^2-3x^2)K_2(\kappa r)\right\}$$

(3.15)

$$\sigma_{yy} = -\frac{\mu b}{2\pi(1-\nu)}\frac{y}{r^4}\left\{(y^2-x^2)-\frac{4}{\kappa^2 r^2}(y^2-3x^2)-2x^2\kappa r K_1(\kappa r)+2(y^2-3x^2)K_2(\kappa r)\right\}$$

(3.16)

$$\sigma_{xy} = \frac{\mu b}{2\pi(1-\nu)}\frac{x}{r^4}\left\{(x^2-y^2)-\frac{4}{\kappa^2 r^2}(x^2-3y^2)-2y^2\kappa r K_1(\kappa r)+2(x^2-3y^2)K_2(\kappa r)\right\}$$

(3.17)

$$\sigma_{zz} = -\frac{\mu b\nu}{\pi(1-\nu)}\frac{y}{r^2}\{1-\kappa r K_1(\kappa r)\}.$$

(3.18)

The stress σ_{zz} satisfies the condition $\sigma_{zz} = \nu(\sigma_{xx}+\sigma_{yy})$. The trace of the stress tensor σ_k^k produced by the edge dislocation in an isotropic medium is

$$\sigma_k^k = -\frac{\mu b(1+\nu)}{\pi(1-\nu)}\frac{y}{r^2}\{1-\kappa r K_1(\kappa r)\}.$$

(3.19)

The spatial distribution of stresses of an edge dislocation near the dislocation line is presented in figure 1.

Let us now discuss some details of the core modification of the stress fields. The stress fields have no artificial singularities at the core and the maximum stress occurs at a short distance away from the dislocation line (see figure 2). In fact, when $r \to 0$, we have

$$K_1(\kappa r) \to \frac{1}{\kappa r} \qquad K_2(\kappa r) \to -\frac{1}{2}+\frac{2}{(\kappa r)^2}$$

(3.20)

and thus $\sigma_{ij} \to 0$. It can be seen that the stresses have the following extreme values: $|\sigma_{xx}(0,y)| \simeq 0.546\kappa\frac{\mu b}{2\pi(1-\nu)}$ at $|y| \simeq 0.996\kappa^{-1}$, $|\sigma_{yy}(0,y)| \simeq 0.260\kappa\frac{\mu b}{2\pi(1-\nu)}$ at $|y| \simeq 1.494\kappa^{-1}$, $|\sigma_{xy}(x,0)| \simeq 0.260\kappa\frac{\mu b}{2\pi(1-\nu)}$ at $|x| \simeq 1.494\kappa^{-1}$ and $|\sigma_{zz}(0,y)| \simeq 0.399\kappa\frac{\mu b\nu}{\pi(1-\nu)}$ at $|y| \simeq 1.114\kappa^{-1}$. The stresses σ_{xx},σ_{yy} and σ_{xy} are modified near the dislocation core ($0 \leqslant r \leqslant 12\kappa^{-1}$). Note that $|\sigma_{xy}(x,0)| \simeq 0.260\kappa\frac{\mu b}{2\pi(1-\nu)}$ can be identified as the theoretical shear strength. The stress σ_{zz} and the trace σ_k^k are modified in the region $0 \leqslant r \leqslant 6\kappa^{-1}$. Far from the dislocation line ($r \gg 12\kappa^{-1}$), the gauge theoretical and the classical solutions of the stress of an edge dislocation coincide. Thus, the characteristic internal length κ^{-1} determines the position and the magnitude of the stress maxima. It is interesting and important to note that the gauge theoretical solutions (3.15)–(3.18) agree precisely with the gradient solutions given by Gutkin and Aifantis [37, 38] (with $\kappa^{-2} = c$, c is the gradient coefficient).

For the elastic strain of an edge dislocation we find

$$E_{xx} = -\frac{b}{4\pi(1-\nu)}\frac{y}{r^2}\left\{(1-2\nu)+\frac{2x^2}{r^2}+\frac{4}{\kappa^2 r^4}(y^2-3x^2)\right.$$
$$\left.-2\left(\frac{y^2}{r^2}-\nu\right)\kappa r K_1(\kappa r)-\frac{2}{r^2}(y^2-3x^2)K_2(\kappa r)\right\}$$

(3.21)

$$E_{yy} = -\frac{b}{4\pi(1-\nu)}\frac{y}{r^2}\left\{(1-2\nu)-\frac{2x^2}{r^2}-\frac{4}{\kappa^2 r^4}(y^2-3x^2)\right.$$
$$\left.-2\left(\frac{x^2}{r^2}-\nu\right)\kappa r K_1(\kappa r)+\frac{2}{r^2}(y^2-3x^2)K_2(\kappa r)\right\}$$
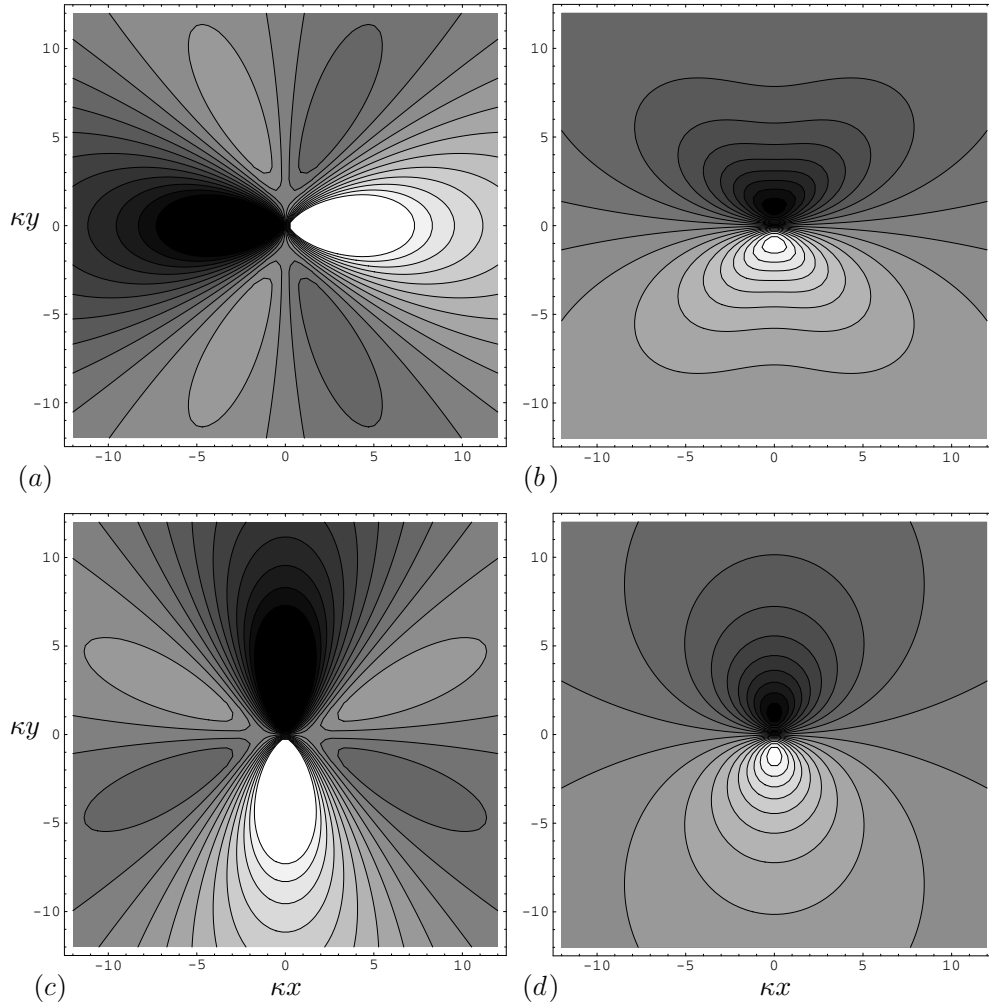
(3.22)

Figure 1. Stress contours of an edge dislocation near the dislocation line: (a) σ_{xy}, (b) σ_{xx}, (c) σ_{yy}, (d) σ_{zz}.

$$E_{xy} = \frac{b}{4\pi(1-\nu)}\frac{x}{r^2}\left\{1 - \frac{2y^2}{r^2} - \frac{4}{\kappa^2 r^4}(x^2 - 3y^2) - \frac{2y^2}{r^2}\kappa r K_1(\kappa r) + \frac{2}{r^2}(x^2 - 3y^2)K_2(\kappa r)\right\}$$

(3.23)

which is in agreement with the solution given by Gutkin and Aifantis [36–38] in the framework of strain gradient elasticity. The plane-strain condition $E_{zz} = 0$ of the classical dislocation theory is also valid in the dislocation core region. The components of the strain tensor have the following extreme values ($\nu = 0.3$): $|E_{xx}(0, y)| \simeq 0.308\kappa\frac{b}{4\pi(1-\nu)}$ at $|y| \simeq 0.922\kappa^{-1}$, $|E_{yy}(0, y)| \simeq 0.010\kappa\frac{b}{4\pi(1-\nu)}$ at $|y| \simeq 0.218\kappa^{-1}$, $|E_{yy}(0, y)| \simeq 0.054\kappa\frac{b}{4\pi(1-\nu)}$ at $|y| \simeq 4.130\kappa^{-1}$ and $|E_{xy}(x, 0)| \simeq 0.260\kappa\frac{b}{4\pi(1-\nu)}$ at $|x| \simeq 1.494\kappa^{-1}$. It is interesting to note that $E_{yy}(0, y)$ is much smaller than $E_{xx}(0, y)$ within the core region (see also [37, 38]). The dilatation $E^k_{\ k}$ reads

$$E^k_{\ k} = -\frac{b(1-2\nu)}{2\pi(1-\nu)}\frac{y}{r^2}\{1 - \kappa r K_1(\kappa r)\}.$$
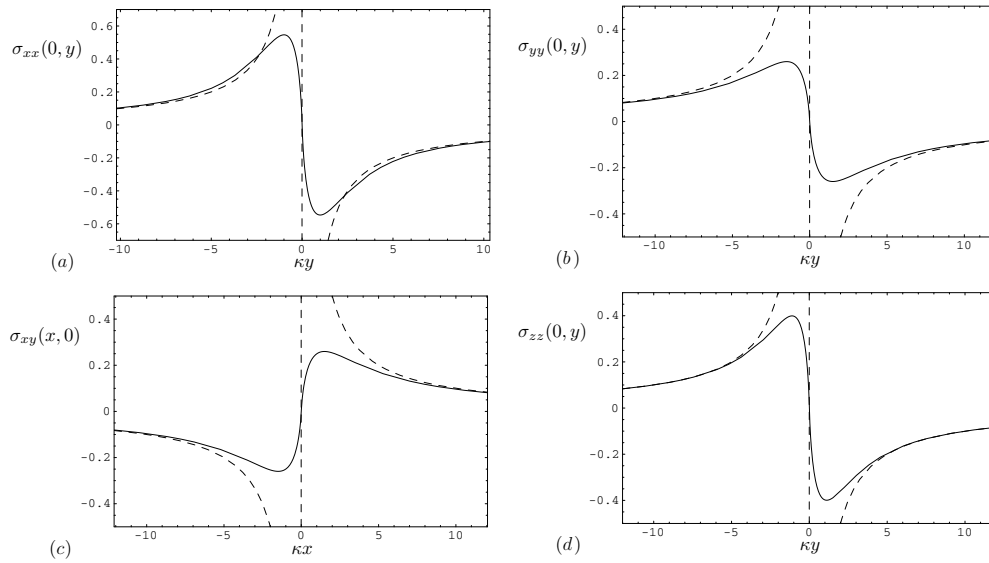
(3.24)

Figure 2. The stress components near the dislocation line: $(a)\,\sigma_{xx}(0,y)$, $(b)\,\sigma_{yy}(0,y)$, $(c)\,\sigma_{xy}(x,0)$ are given in units of $\mu b\kappa/[2\pi(1-\nu)]$ and $(d)\,\sigma_{zz}(0,y)$ is given in units of $\mu b\nu\kappa/[\pi(1-\nu)]$. The dashed curves represent the classical stress components.

It is convenient to rewrite the stress and strain fields in cylindrical coordinates. The stress function reads in cylindrical coordinates:

$$f = -\frac{\mu b}{2\pi(1-\nu)}\sin\varphi\left\{r\ln r + \frac{2}{\kappa^2 r}(1 - \kappa r K_1(\kappa r))\right\}. \tag{3.25}$$

Here the stress function is related to the stresses by the equations

$$\sigma_{rr} = \frac{1}{r}\partial_r f + \frac{1}{r^2}\partial_{\varphi\varphi}^2 f \qquad \sigma_{\varphi\varphi} = \partial_{rr}^2 f \qquad \sigma_{r\varphi} = -\partial_r\left(\frac{1}{r}\partial_\varphi f\right) \qquad \sigma_{zz} = \nu(\sigma_{rr} + \sigma_{\varphi\varphi}). \tag{3.26}$$

In this way we find the stress of a straight edge dislocation in cylindrical coordinates

$$\sigma_{rr} = -\frac{\mu b}{2\pi(1-\nu)}\frac{\sin\varphi}{r}\left\{1 - \frac{4}{\kappa^2 r^2} + 2K_2(\kappa r)\right\} \tag{3.27}$$

$$\sigma_{r\varphi} = \frac{\mu b}{2\pi(1-\nu)}\frac{\cos\varphi}{r}\left\{1 - \frac{4}{\kappa^2 r^2} + 2K_2(\kappa r)\right\} \tag{3.28}$$

$$\sigma_{\varphi\varphi} = -\frac{\mu b}{2\pi(1-\nu)}\frac{\sin\varphi}{r}\left\{1 + \frac{4}{\kappa^2 r^2} - 2K_2(\kappa r) - 2\kappa r K_1(\kappa r)\right\} \tag{3.29}$$

$$\sigma_{zz} = -\frac{\mu b\nu}{\pi(1-\nu)}\frac{\sin\varphi}{r}\{1 - \kappa r K_1(\kappa r)\} \tag{3.30}$$

and the trace of the stress tensor reads

$$\sigma_k^k = -\frac{\mu b(1+\nu)}{\pi(1-\nu)}\frac{\sin\varphi}{r}\{1 - \kappa r K_1(\kappa r)\}. \tag{3.31}$$

Eventually, the elastic strain of a straight edge dislocation is given in cylindrical coordinates as follows:

$$E_{rr} = -\frac{b}{4\pi(1-v)}\frac{\sin\varphi}{r}\left\{(1-2v) - \frac{4}{\kappa^2 r^2} + 2K_2(\kappa r) + 2v\kappa r K_1(\kappa r)\right\} \tag{3.32}$$

$$E_{r\varphi} = \frac{b}{4\pi(1-v)}\frac{\cos\varphi}{r}\left\{1 - \frac{4}{\kappa^2 r^2} + 2K_2(\kappa r)\right\} \tag{3.33}$$

$$E_{\varphi\varphi} = -\frac{b}{4\pi(1-v)}\frac{\sin\varphi}{r}\left\{(1-2v) + \frac{4}{\kappa^2 r^2} - 2K_2(\kappa r) - 2(1-v)\kappa r K_1(\kappa r)\right\}. \tag{3.34}$$

The dilatation reads

$$E^k_{k} = -\frac{b(1-2v)}{2\pi(1-v)}\frac{\sin\varphi}{r}\{1 - \kappa r K_1(\kappa r)\}. \tag{3.35}$$

Let us now calculate the distortion of an edge dislocation. The distortion β^a_{i} is given in terms of the stress function (3.14):

$$\beta^a_{i} = \frac{1}{2\mu}\begin{pmatrix} \partial^2_{yy}f - v\Delta f & -\partial^2_{xy}f + 2\mu\omega & 0 \\ -\partial^2_{xy}f - 2\mu\omega & \partial^2_{xx}f - v\Delta f & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{3.36}$$

where ω is used to express the antisymmetric part of the distortion, $\omega \equiv \beta_{[xy]}$. Eventually, ω is determined from the conditions

$$T^x_{xy} = \frac{1}{2\mu}(2\mu\partial_x\omega - (1-v)\partial_y\Delta f) \tag{3.37}$$

$$T^y_{xy} = \frac{1}{2\mu}(2\mu\partial_y\omega + (1-v)\partial_x\Delta f) \equiv 0. \tag{3.38}$$

We find for the elastic distortion of the straight dislocation

$$\beta^x_{x} = -\frac{b}{4\pi(1-v)}\frac{y}{r^2}\left\{(1-2v) + \frac{2x^2}{r^2} + \frac{4}{\kappa^2 r^4}(y^2 - 3x^2)\right.$$
$$\left. - 2\left(\frac{y^2}{r^2} - v\right)\kappa r K_1(\kappa r) - \frac{2}{r^2}(y^2 - 3x^2)K_2(\kappa r)\right\} \tag{3.39}$$

$$\beta^x_{y} = \frac{b}{4\pi(1-v)}\frac{x}{r^2}\left\{(3-2v) - \frac{2y^2}{r^2} - \frac{4}{\kappa^2 r^4}(x^2 - 3y^2)\right.$$
$$\left. - 2\left((1-v) + \frac{y^2}{r^2}\right)\kappa r K_1(\kappa r) + \frac{2}{r^2}(x^2 - 3y^2)K_2(\kappa r)\right\} \tag{3.40}$$

$$\beta^y_{x} = -\frac{b}{4\pi(1-v)}\frac{x}{r^2}\left\{(1-2v) + \frac{2y^2}{r^2} + \frac{4}{\kappa^2 r^4}(x^2 - 3y^2)\right.$$
$$\left. - 2\left((1-v) - \frac{y^2}{r^2}\right)\kappa r K_1(\kappa r) - \frac{2}{r^2}(x^2 - 3y^2)K_2(\kappa r)\right\} \tag{3.41}$$

$$\beta^y_{y} = -\frac{b}{4\pi(1-v)}\frac{y}{r^2}\left\{(1-2v) - \frac{2x^2}{r^2} - \frac{4}{\kappa^2 r^4}(y^2 - 3x^2)\right.$$
$$\left. - 2\left(\frac{x^2}{r^2} - v\right)\kappa r K_1(\kappa r) + \frac{2}{r^2}(y^2 - 3x^2)K_2(\kappa r)\right\} \tag{3.42}$$

and for the rotation

$$\omega_z \equiv -\omega = -\frac{b}{2\pi}\frac{x}{r^2}\{1 - \kappa r K_1(\kappa r)\}. \tag{3.43}$$

Equation (3.43) is in agreement with the rotation vector calculated in the linear theory of dislocations in the Cosserat continuum [50]. The far fields of (3.39)–(3.43) are identical to the classical ones given in [74].

We now rewrite the distortions (3.39)–(3.41) as follows:

$$\beta^x = \frac{b}{2\pi}\left\{\left((1 - \kappa r K_1(\kappa r)) + \frac{1}{2(1-\nu)}\cos 2\varphi\left(1 - \frac{4}{\kappa^2 r^2} + 2K_2(\kappa r)\right)\right)\mathrm{d}\varphi\right.$$
$$\left. + \frac{1}{2(1-\nu)}\sin 2\varphi\left(\frac{4}{\kappa^2 r^3} - \kappa K_1(\kappa r) - \frac{2}{r}K_2(\kappa r)\right)\mathrm{d}r\right\} \equiv \mathrm{d}u^x + \phi^x. \tag{3.44}$$

We may interpret the field

$$\phi^x = -\frac{b\kappa^2}{2\pi}\varphi r K_0(\kappa r)\,\mathrm{d}r \tag{3.45}$$

as the proper incompatible part (negative plastic distortion). This proper incompatible distortion is exactly the same as ϕ^z of a screw dislocation (see [2, 3]). In components it reads

$$\phi^x_x = -\frac{b\kappa^2}{2\pi}x\varphi K_0(\kappa r) \qquad \phi^x_y = -\frac{b\kappa^2}{2\pi}y\varphi K_0(\kappa r). \tag{3.46}$$

The appearance of such proper incompatible distortion is a typical result of the gauge theory of dislocations. When $\kappa^{-1} \to 0$, the incompatible distortion (3.46) converts to $\phi^x_x = -bx\varphi\delta(r)$ and $\phi^x_y = -by\varphi\delta(r)$. The compatible part of equation (3.44) is given in terms of the displacement field

$$u^x = \frac{b}{2\pi}\left\{(1 - \kappa r K_1(\kappa r))\varphi + \frac{1}{4(1-\nu)}\sin 2\varphi\left(1 - \frac{4}{\kappa^2 r^2} + 2K_2(\kappa r)\right)\right\}. \tag{3.47}$$

It is a modified displacement field (see figure 3(a)). We used $\varphi = \arctan(y/x)$ as a multi-valued field. Note that the first part in (3.47) is different from the corresponding one given in gradient elasticity by Gutkin and Aifantis [36, 37]. One reason is that they used a compatible elastic distortion. It is important to bear in mind that u^x is not a simple arctan function as in the Volterra model of a dislocation. Additionally, we find from the distortion

$$\beta^y = -\frac{b}{4\pi(1-\nu)}\left\{\sin 2\varphi\left(1 - \frac{4}{\kappa^2 r^2} + 2K_2(\kappa r)\right)\mathrm{d}\varphi - \left((1-2\nu)\left(\frac{1}{r} - \kappa K_1(\kappa r)\right)\right.\right.$$
$$\left.\left. + \cos 2\varphi\left(\frac{4}{\kappa^2 r^3} - \kappa K_1(\kappa r) - \frac{2}{r}K_2(\kappa r)\right)\right)\mathrm{d}r\right\} \equiv \mathrm{d}u^y + \phi^y \tag{3.48}$$

the incompatible part

$$\phi^y = 0 \tag{3.49}$$

and a single-valued displacement field

$$u^y = -\frac{b}{8\pi(1-\nu)}\left\{2(1-2\nu)(\ln r + K_0(\kappa r)) + \cos 2\varphi\left(1 - \frac{4}{\kappa^2 r^2} + 2K_2(\kappa r)\right)\right\} \tag{3.50}$$

which has no singularity (see figure 3(b)). This means that β^y is a proper compatible distortion. This is the reason why the field u^y agrees precisely with the corresponding formula in gradient elasticity (see [36, 37]). The displacement field u^y differs from the classical one in the region $0 \leqslant r \leqslant 12\kappa^{-1}$. In this framework the displacement fields (3.47) and (3.50) have
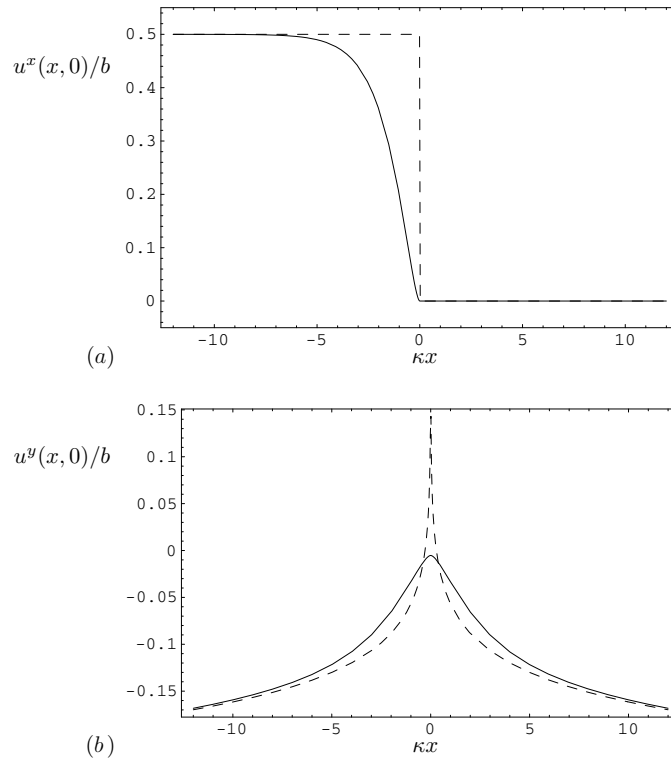
Figure 3. Effective displacement fields ($v = 0.3$): (a) $u^x(x, y \to +0)/b$, (b) $u^y(x, 0)/b$. The dashed curves represent the classical solution.

no singularity. The far fields of the displacements (3.47) and (3.50) are identical to the classical ones given in [26, 75, 76]. It can be seen that solution (3.47) leads to an asymmetric smoothing of the displacement profile (see figure 3(a)). In fact, it is asymmetric around the dislocation line. Note that the incompatible distortion (3.46) also has an asymmetric form. These asymmetries would lead to an unexpected asymmetry of the dislocation core. From the mathematical point of view, the asymmetry is induced by the decomposition of the elastic distortion (2.2) into compatible and incompatible parts. Keep in mind that such a decomposition is gauge invariant (see equation (2.3)) and we have a local translation as an additional degree of freedom. This gauge invariance can be used to improve the situation. So far we have considered φ as a multi-valued field. On the other hand, in the defect theory [35–38, 74] φ is usually used as single-valued and discontinuous function. It is made unique by cutting the half-plane $y = 0$ at $x < 0$ and assuming φ to jump from π to $-\pi$ when crossing the cut. If we use the single-valued discontinuous form for φ and the local gauge transformation (2.3) with the gauge function

$$\tau^x = \frac{b}{2\pi} \operatorname{sign}(y) \frac{\pi}{2} \kappa r K_1(\kappa r) \tag{3.51}$$

we obtain for the improved displacement field

$$u^x = \frac{b}{2\pi} \left\{ \varphi(1 - \kappa r K_1(\kappa r)) + \frac{\pi}{2} \operatorname{sign}(y) \kappa r K_1(\kappa r) + \frac{1}{2(1 - v)} \frac{xy}{r^2} \left(1 - \frac{4}{\kappa^2 r^2} + 2K_2(\kappa r) \right) \right\} \tag{3.52}$$
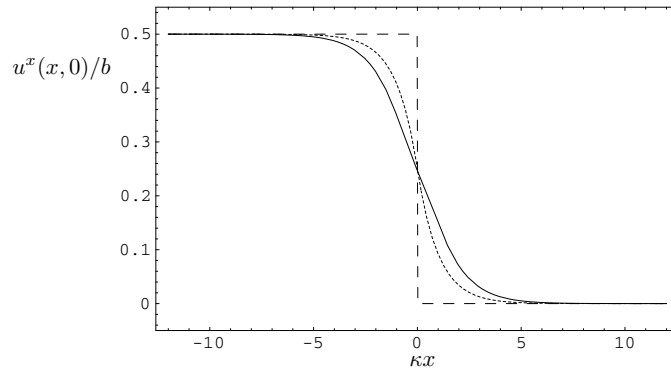
Figure 4. Displacement $u^x(x, y \rightarrow +0)/b$ near the line of edge dislocation within gauge model (solid), the gradient model (small dashed) and the classical elasticity (dashed). We used $\kappa^2 = 1/c$.

and for the proper incompatible distortion

$$\phi^x_{\ x} = -\frac{b}{2\pi}\kappa^2 x K_0(\kappa r)\left(\varphi - \frac{\pi}{2}\,\mathrm{sign}(y)\right) \tag{3.53}$$

$$\phi^x_{\ y} = -\frac{b}{2\pi}\left\{\kappa^2 y K_0(\kappa r)\left(\varphi - \frac{\pi}{2}\,\mathrm{sign}(y)\right) + \pi\delta(y)(1 - \mathrm{sign}(x)[1 - \kappa r K_1(\kappa r)])\right\}. \tag{3.54}$$

Now the displacements and the incompatible distortions are symmetric in the core region around the dislocation line[8]. It is interesting to compare the displacement field (3.52) with the displacement calculated by Gutkin and Aifantis. When $y \rightarrow 0$, the Bessel function terms in (3.52) lead to the symmetric smoothing of the displacement profile in contrast to the abrupt jump occurring in the profile of the classical solution (see figure 4). When $y \rightarrow 0$, the displacement $u^x(x, y)$ calculated by Gutkin and Aifantis [36, 38] within the gradient model has the explicit form

$$u^x(x, y \rightarrow 0) = \frac{b}{2\pi}\,\mathrm{sign}(y)\frac{\pi}{2}\left\{1 - \mathrm{sign}(x)\left(1 - e^{-|x|/\sqrt{c}}\right)\right\}. \tag{3.55}$$

The exponential term which appears in the gradient solution (3.55) leads to the smoothing of the displacement profile (see figure 4). The size of the transition zone is approximately $12\kappa^{-1}$ which gives the value $r_c \simeq 6\kappa^{-1}$ for the core radius. Consequently, in the gauge theory of dislocations and gradient elasticity the dislocation core appears naturally. Therefore, these displacement fields may be used to model the dislocation core. In this way, one can compare the calculated dislocation core with HRTEM micrographs and related computer simulations of the core region. It would be very interesting to check a possible material dependence of κ^{-1}.

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[8] Note that in the case of a screw dislocation the symmetrical displacement and the corresponding incompatible distortion read, respectively,

$$u^z = \frac{b}{2\pi}\left\{\varphi(1 - \kappa r K_1(\kappa r)) + \frac{\pi}{2}\,\mathrm{sign}(y)\kappa r K_1(\kappa r)\right\}$$

and

$$\phi^z_{\ x} = -\frac{b}{2\pi}\kappa^2 x K_0(\kappa r)\left(\varphi - \frac{\pi}{2}\,\mathrm{sign}(y)\right)$$

$$\phi^z_{\ y} = -\frac{b}{2\pi}\left\{\kappa^2 y K_0(\kappa r)\left(\varphi - \frac{\pi}{2}\,\mathrm{sign}(y)\right) + \pi\delta(y)(1 - \mathrm{sign}(x)[1 - \kappa r K_1(\kappa r)])\right\}.$$
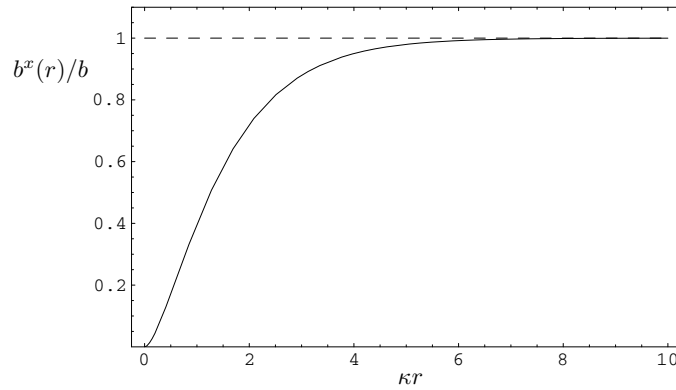
Figure 5. Effective Burgers vector $b^x(r)/b$ (solid).

Finally, the effective Burgers vector can be calculated as

$$b^x(r) = \oint_\gamma \beta^x = b\{1 - \kappa r K_1(\kappa r)\}. \tag{3.56}$$

This effective Burgers vector $b^x(r)$ differs from the constant Burgers vector b in the region $0 \leqslant r \leqslant 6\kappa^{-1}$ (see figure 5). For the value of $\kappa^{-1} = 0.25a$, the core radius is $r_c = 1.5a$ and for the value of $\kappa^{-1} = 0.399a$, the core radius is $r_c = 2.4a$. Note that the effective Burgers vector $b^x(r)$ of an edge dislocation has the same form as the effective Burgers vector $b^z(r)$ of a screw dislocation which is obtained in [2, 4, 9, 10].

3.2. The dislocation density, moment stress and core energy

The proper incompatible part of the elastic distortion gives rise to a localized torsion and moment stress tensor. We find for the torsion

$$T^x_{xy} = -T^x_{yx} = \frac{b\kappa^2}{2\pi} K_0(\kappa r) \tag{3.57}$$

and the dislocation density of an edge dislocation

$$\alpha^{xz} = \frac{b\kappa^2}{2\pi} K_0(\kappa r) \tag{3.58}$$

which satisfies the translational Bianchi identity: $\partial_i \alpha^{ai} = 0$. On can see that the dislocation density is short reaching. It is interesting to note that the torsion T^x_{xy} of an edge dislocation has the same form as the torsion T^z_{xy} of a screw dislocation which is given in [2–4, 9, 10]. In the limit $\kappa^{-1} \to 0$, the gauge theoretical result (3.58) converts to the classical dislocation density $\alpha^{xz} = b\delta(r)$. Additionally, the dislocation density (3.58) agrees with Eringen's two-dimensional nonlocal modulus (nonlocal kernel) used in [28–31]. Consequently, the dislocation density tensor is Green's function of the Helmholtz equation:

$$(1 - \kappa^{-2}\Delta)\alpha^{xz}(r) = b\delta(r). \tag{3.59}$$

This torsion gives rise to a localized moment stress. The localized moment stress of bending type is given with the help of (2.27) as

$$H_{xz} = \frac{\mu b}{\pi(1 - \nu)} K_0(\kappa r) \qquad H_{zx} = -\frac{\mu b\nu}{\pi(1 - \nu)} K_0(\kappa r). \tag{3.60}$$

This expression is very close to the moment stress of an edge dislocation given in [77]. By means of the dislocation density and the moments stress we are able to calculate the core energy of an edge dislocation. The dislocation core energy density is given by

$$V_{\text{core}} = \frac{1}{2}\alpha^{ai}H_{ai} = \frac{\mu b^2 \kappa^2}{4\pi^2(1-\nu)}K_0(\kappa r)^2. \tag{3.61}$$

Obviously, the moment stress H_{zx} gives no contribution to the dislocation core energy. We obtain for the dislocation core energy per unit length of the straight edge dislocation

$$\begin{aligned}
E_{\text{core}} &= \int_0^{2\pi} d\varphi \int_0^R r V_{\text{core}} \, dr \\
&= \frac{\mu b^2 \kappa^2}{4\pi(1-\nu)} r^2 \{K_0(\kappa r)^2 - K_1(\kappa r)^2\}|_0^R \\
&= \frac{\mu b^2}{4\pi(1-\nu)} \{1 + \kappa^2 R^2 (K_0(\kappa R)^2 - K_1(\kappa R)^2)\}
\end{aligned} \tag{3.62}$$

where R denotes the size of the solid body. In the limit $R \to \infty$, the core energy of the straight edge dislocation,

$$E_{\text{core}} = \frac{\mu b^2}{4\pi(1-\nu)} \tag{3.63}$$

coincides with the core energy that is calculated for the edge dislocation in the Peierls–Nabarro model [26]. The core energy is different from the constant value in the region $0 \leqslant \kappa R \leqslant 3$.

The far-reaching rotation gradients read

$$\begin{aligned}
k^x_{xy} &= -k^x_{yx} = -\frac{b}{2\pi r^4}\{(x^2 - y^2)(1 - \kappa r K_1(\kappa r)) - \kappa^2 x^2 r^2 K_0(\kappa r)\} \\
k^y_{xy} &= -k^y_{yx} = -\frac{b}{2\pi r^4}xy\{2(1 - \kappa r K_1(\kappa r)) - \kappa^2 r^2 K_0(\kappa r)\}.
\end{aligned} \tag{3.64}$$

The form of these rotation gradients is in agreement with the expressions calculated within the theory of Cosserat media (see [48, 51]).

3.3. The straight edge dislocation in a cylinder

Now we consider a straight edge dislocation in a cylinder of finite outer radius R with an outer boundary free from external forces. An advantage in our framework is that we do not need an inner radius because the stress and strain fields are zero at $r = 0$ in quite a natural way. If the outer boundary is to be free, it is necessary to add to the stress function (3.5) an additional stress function $\chi' = ACyr^2$. Then the modified stress function is of the form

$$f = -\frac{\mu b}{2\pi(1-\nu)}\sin\varphi\left\{r\ln r + \frac{2}{\kappa^2 r}(1 - \kappa r K_1(\kappa r) - 4Cr^2) - Cr^3\right\} \tag{3.65}$$

where the constant C is determined by the semi-classical boundary condition that the stresses σ_{rr} and $\sigma_{r\varphi}$ should be zero at $r = R$. In this way we find the stress of the edge dislocation in cylindrical coordinates

$$\sigma_{rr} = -\frac{\mu b}{2\pi(1-\nu)}\frac{\sin\varphi}{r}\left\{1 - \frac{4}{\kappa^2 r^2} + 2K_2(\kappa r) - 2Cr^2\right\} \tag{3.66}$$

$$\sigma_{r\varphi} = \frac{\mu b}{2\pi(1-\nu)}\frac{\cos\varphi}{r}\left\{1 - \frac{4}{\kappa^2 r^2} + 2K_2(\kappa r) - 2Cr^2\right\} \tag{3.67}$$

$$\sigma_{\varphi\varphi} = -\frac{\mu b}{2\pi(1-\nu)} \frac{\sin\varphi}{r} \left\{ 1 + \frac{4}{\kappa^2 r^2} - 2K_2(\kappa r) - 2\kappa r K_1(\kappa r) - 6Cr^2 \right\} \tag{3.68}$$

$$\sigma_{zz} = -\frac{\mu b \nu}{\pi(1-\nu)} \frac{\sin\varphi}{r} \{ 1 - \kappa r K_1(\kappa r) - 4Cr^2 \} \tag{3.69}$$

with

$$C = \frac{1}{2R^2} \left\{ 1 - \frac{4}{\kappa^2 R^2} + 2K_2(\kappa R) \right\}. \tag{3.70}$$

The corresponding semi-classical boundary is given by

$$\sigma^k_{\ k} = -\frac{\mu b(1+\nu)}{\pi(1-\nu)} \frac{\sin\varphi}{r} \{ 1 - \kappa r K_1(\kappa r) - 4Cr^2 \}. \tag{3.71}$$

The elastic strain of this edge dislocation in a cylinder is

$$E_{rr} = -\frac{b}{4\pi(1-\nu)} \frac{\sin\varphi}{r} \left\{ (1-2\nu) - \frac{4}{\kappa^2 r^2} + 2K_2(\kappa r) + 2\nu\kappa r K_1(\kappa r) - (2-8\nu)Cr^2 \right\} \tag{3.72}$$

$$E_{r\varphi} = \frac{b}{4\pi(1-\nu)} \frac{\cos\varphi}{r} \left\{ 1 - \frac{4}{\kappa^2 r^2} + 2K_2(\kappa r) - 2Cr^2 \right\} \tag{3.73}$$

$$E_{\varphi\varphi} = -\frac{b}{4\pi(1-\nu)} \frac{\sin\varphi}{r} \left\{ (1-2\nu) + \frac{4}{\kappa^2 r^2} - 2K_2(\kappa r) - 2(1-\nu)\kappa r K_1(\kappa r) - (6-8\nu)Cr^2 \right\}. \tag{3.74}$$

The corresponding dilatation field reads

$$E^k_{\ k} = -\frac{b(1-2\nu)}{2\pi(1-\nu)} \frac{\sin\varphi}{r} \{ 1 - \kappa r K_1(\kappa r) - 4Cr^2 \}. \tag{3.75}$$

From condition (3.38) we find for the rotation of an edge dislocation in a cylinder

$$\omega_z \equiv -\omega = -\frac{b}{2\pi} \frac{\cos\varphi}{r} \{ 1 - \kappa r K_1(\kappa r) + 4Cr^2 \} \tag{3.76}$$

which gives an additional constant contribution to the bend–twist tensor k^x_{xy} in equation (3.64). The dislocation density of the edge dislocation in a cylinder is determined by condition (3.37) and has the same form as the dislocation in an infinitely extended body (3.58).

For convenience we give the displacement components in Cartesian coordinates

$$u^x = \frac{b}{2\pi} \left\{ \varphi(1 - \kappa r K_1(\kappa r)) + \frac{\pi}{2} \text{sign}(y)\kappa r K_1(\kappa r) \right.$$
$$\left. + \frac{1}{2(1-\nu)} \frac{xy}{r^2} \left(1 - \frac{4}{\kappa^2 r^2} + 2K_2(\kappa r) \right) + \frac{C(3-4\nu)}{(1-\nu)} xy \right\} \tag{3.77}$$

$$u^y = -\frac{b}{4\pi(1-\nu)} \left\{ (1-2\nu)(\ln r + K_0(\kappa r)) + \frac{x^2 - y^2}{2r^2} \left(1 - \frac{4}{\kappa^2 r^2} + 2K_2(\kappa r) \right) \right.$$
$$\left. + C(x^2(5-4\nu) - y^2(1-4\nu)) \right\}. \tag{3.78}$$

Note that this boundary-value problem of the edge dislocation may be applied to describe dislocation behaviour in nanoparticles and nanowires.

4. The relation to the nonlocal theory of dislocations

Let me now give the relation to Eringen's so-called nonlocal theory of dislocations [28, 30, 31]. It includes the effect of long-range interatomic forces so that it can be used as a continuum model of the atomic lattice dynamics. Using Green's function of the Helmholtz equation (3.59) we may solve the field equation for every component of the stress field (2.46) with the help of the convolution integral

$$\sigma_{ij}(r) = \int_V G(r - r') \overset{\circ}{\sigma}_{ij}(r') \, dv(r') \tag{4.1}$$

with

$$G(r - r') \equiv \frac{1}{b} \alpha^{xz}(r - r') = \frac{\kappa^2}{2\pi} K_0(\kappa(r - r')). \tag{4.2}$$

In this way we deduce Eringen's so-called nonlocal constitutive relation for a linear homogeneous, isotropic solid. Therefore, the gauge theoretical description of dislocations in this paper is close to the nonlocal elasticity of dislocations [28, 30, 31], if one uses the special two-dimensional Green's function (4.2). The form of the nonlocal kernel (dislocation density) is fixed by the incompatible distortion. Finally, we may conclude that the gauge theory of dislocation contains a weak nonlocality due to the moment stresses in the dislocation core region. This observation is in agreement with Kröner's opinion that the introduction of moment stresses represents the first step in a transition from local to a nonlocal elasticity theory [77–80].

The characteristic internal length in these theories is κ^{-1}. This length may be selected to be proportional to the lattice parameter a for a single crystal, i.e.

$$\kappa^{-1} = e_0 a \tag{4.3}$$

where e_0 is a non-dimensional constant or material function which can be determined by one experiment [30]. For $e_0 = 0$ we have classical elasticity and there are no effects of moment stresses. In this limit the nonlocal kernel must revert to the Dirac delta measure. In [28, 30], the choice $e_0 = 0.399$ and in [32–34], the choice $e_0 = 0.25$ are proposed (see also [2, 3, 38]).

Consequently, we can conclude that the gauge theoretical solution of the stress field for an edge dislocation given in this paper is also a solution in nonlocal elasticity. However, a difference to our approach is that Eringen's nonlocal theory works with compatible strain instead of incompatible strain and the strains and displacements have the classical form with singularities at the dislocation line.

5. Discussion and conclusion

The translational gauge theory of dislocations has been employed to consider the straight edge dislocation in an infinitely extended body and in a cylinder. This translational gauge theory unifies elasticity and plasticity to elastoplasticity in field theoretical way. In this framework, elastoplasticity is a theory with force stresses and moment stresses. The size of this moment stress can be calculated even from linearized elastoplasticity theory. Using the stress function method, exact analytical solutions for the displacements, strain and stress fields have been derived which do not have any artificial divergency at the dislocation core and the corresponding far fields agree with the classical ones for the edge dislocation. The strain and stress fields of an edge dislocation in an infinitely extended body are in exact agreement with those obtained by Gutkin and Aifantis [36, 37] through an analysis of the edge dislocation in the theory of gradient elasticity. Nevertheless, the gauge theoretical and the gradient theoretical approaches are different and they give slightly different results, e.g., the displacement field u^x

due to the incompatible distortion in the gauge theory of dislocations. On the other hand, the strain gradient elasticity used a purely compatible distortion. However, the difference cannot be too big and the numerical values should be of the same order. The dislocation density of an edge dislocation obtained in the gauge theoretical framework is in agreement with the nonlocal kernel given by Ari and Eringen [28–31]. From this point of view, the nonlocal elasticity theory approach is close to ours. On the other hand, Eringen's nonlocal elasticity contains compatible strains only, and the strain and displacement fields have the same form as in classical elasticity. In the gauge theoretical analysis, the dislocation core arises naturally. We have calculated the localized moment stress which is the response to dislocation density and the far-reaching rotation gradient (bend–twist tensor).

The dislocation model used in this paper contains, in general, ten material constants. The total Lagrangian of this model is given in terms of the distortion, the dislocation density (torsion tensor) and deWit's bend–twist tensor. With the help of some special choices (2.24), (2.40) and (2.43), we reduced the number of constants to three material constants, namely two elastic constants and one internal length. As a result, we obtain a dislocation theory with symmetric force stresses. The special choice (2.24) for the three coefficients a_1, a_2 and a_3 in the dislocation gauge Lagrangian gives the core energy of the edge dislocation in an infinite medium which is in agreement with the expression obtained in the Peierls–Nabarro model. It is important to note that one can also use the total Lagrangian of this dislocation model for the gauge theoretical description of screw dislocations. In this way, one reproduces all quantities of the screw dislocation calculated in [2]. Therefore, by using the Lagrangian proposed in this paper, one can obtain gauge theoretical solutions for the edge and screw dislocations which have no singularities and agree with the classical solutions in the far field. The (linear) gauge equation formulated in terms of the stress field is an inhomogeneous Helmholtz equation for the screw and edge dislocations.

It will be interesting to apply the nonsingular solutions of the edge and screw dislocations to dislocation modelling. One can use the gauge theoretical displacement fields to simulate the core region for HRTEM micrographs. Another interesting application could be the x-ray diffraction profile analysis. One should use the materials tungsten, diamond or aluminium because they are nearly isotropic. In this way the coefficient κ^{-1} should be experimentally verified. In addition, it would be of interest to compare experimental results with the values from different theoretical approaches (gauge theory, strain gradient elasticity and/or nonlocal elasticity). Another possibility of determining the coefficient κ^{-1} should be a lattice-theoretical approach. The coefficient κ^{-1} expresses the interactions and the properties in the core, which are different from those in the undeformed crystal. Consequently, the elastic moduli say nothing about the interactions in the core of dislocations. It should be possible to calculate κ^{-1} from the coupling parameters of the lattice at the dislocation core.

Acknowledgments

References

[1] Lazar M 2000 *Ann. Phys., Lpz* **9** 461
[2] Lazar M 2002 *J. Phys. A: Math. Gen.* **35** 1983

[3]   Lazar M 2002 *Ann. Phys., Lpz* **11** 635
[4]   Malyshev C 2000 *Ann. Phys., NY* **286** 249
[5]   Golebiewska-Lasota A A 1979 *Int. J. Eng. Sci.* **17** 329
[6]   Edelen D G B 1982 *Int. J. Eng. Sci.* **20** 49
[7]   Kadić A and Edelen D G B 1983 *A Gauge Theory of Dislocations and Disclinations (Lecture Notes in Physics vol 174)* (Berlin: Springer)
[8]   Edelen D G B and Lagoudas D C 1988 *Gauge Theory and Defects in Solids* (Amsterdam: North-Holland)
[9]   Valsakumar M C and Sahoo D 1988 *Bull. Mater. Sci.* **10** 3
[10] Edelen D G B 1996 *Int. J. Eng. Sci.* **34** 81
[11] Popov V L 1992 *Int. J. Eng. Sci.* **30** 329
[12] Kadić-Galeb A and Batra R C 1994 *Int. J. Eng. Sci.* **32** 291
[13] Katanaev M O and Volovich I V 1992 *Ann. Phys., NY* **216** 1
[14] Gairola B K D 1981 Gauge invariant formulation of continuum theory of defects *Continuum Models and Discrete Systems Proc. 4th Int. Symp.* ed O Brulin and R K T Hsieh (Amsterdam: North-Holland) p 55
[15] Gairola B K D 1993 Gauge theory of dislocations *continuum Models and Discrete Systems Proc. 7th Int. Symp. (Paderborn, Germany)* ed K-H Anthony and H-J Wagner (Aedermannsdorf, CH: Trans. Techn. Publ.) p 579
[16] Kröner E 1993 A variational principle in nonlinear dislocations theory *Proc. 2nd Int. Conf. Non-Linear Mech.* ed Wei-zang Chien (Beijing: Peking University Press) p 59
[17] Kröner E 1996 Dislocation theory as a physical field theory *Continuum Models and Discrete Systems, Proc. 8th Int. Symp. (Varna, Bulgaria)* ed K Z Markov (Singapore: World Scientific) p 522
[18] Kunin I A 1985 On the gauge theory of dislocations *The Mechanics of Dislocations* ed E C Aifantis and J P Hirth (Metals Park, OH: American Society of Metals) p 69
[19] Kleinert H 1989 *Gauge Fields in Condensed Matter: Vol II. Stresses and Defects* (Singapore: World Scientific)
[20] Osipov V A 1991 *J. Phys. A: Math. Gen.* **24** 3237
[21] Volkov N B 1997 *J. Phys. A: Math. Gen.* **30** 6391
[22] Bogatov N M 2001 *Phys. Status. Solidi* b **228** 651
[23] Peierls R E 1940 *Proc. Phys. Soc. (London)* **54** 34
[24] Nabarro F R N 1947 *Proc. Phys. Soc. (London)* **59** 256
[25] Nabarro F R N 1987 *Theory of Crystal Dislocations* (New York: Dover)
[26] Hirth J P and Lothe J 1982 *Theory of Dislocations* (New York: Wiley)
[27] Eringen A C 1977 *Int. J. Eng. Sci.* **15** 177
[28] Eringen A C 1983 *J. Appl. Phys.* **54** 4703
[29] Ari N and Eringen A C 1983 *Cryst. Lattice Defects Amorphous. Mater.* **10** 33
[30] Eringen A C 1985 Nonlocal continuum theory for dislocaions and fracture *The Mechanics of Dislocations* ed E C Aifantis and J P Hirth (Metals Park, OH: American Society of Metals) p 101
[31] Eringen A C 1987 *Res. Mech.* **21** 313
[32] Altan B S and Aifantis E C 1992 *Scr. Metall. Mater.* **26** 319
[33] Aifantis E C 1994 *J. Mech. Behav. Mater.* **5** 355
[34] Altan B S and Aifantis E C 1997 *J. Mech. Behav. Mater.* **8** 231
[35] Gutkin M Yu and Aifantis E C 1996 *Scr. Mater.* **35** 1353
[36] Gutkin M Yu and Aifantis E C 1997 *Scr. Mater.* **36** 129
[37] Gutkin M Yu and Aifantis E C 1999 *Scr. Mater.* **40** 559
[38] Gutkin M Yu 2000 *Rev. Adv. Mater. Sci.* **1** 27
[39] Brailsford A D 1966 *Phys. Rev.* **142** 383
[40] Kunin I A 1986 *Theory of Elastic Media with Microstructure* (Berlin: Springer)
[41] Vörös G and Kovács I 1993 *Phys. Status Solidi* b **178** 99
[42] Vörös G and Kovács I 1995 *Phil. Mag.* A **72** 949
[43] Günther W 1958 Zur Statik und Kinematik des Cosseratschen Kontinuums *Abh. Braunschw. Wiss. Ges.* **10** 195
[44] Schaefer H 1967 *Z. Angew. Math. Mech.* **47** 485
[45] Kluge G 1969 *Wiss. Z. Tech. Hochsch. Otto van Guericke Magdeb.* **13** 377
[46] Mişicu M 1965 *Rev. Roum. Sci. Tech., Ser. Mec. Appl.* **10** 35
[47] Teodosiu C 1965 *Rev. Roum. Sci. Tech., Ser. Mec. Appl.* **10** 1461
[48] Kessel S 1970 *Z. Angew. Math. Mech.* **50** 547
[49] Knésl Z and Semela F 1972 *Int. J. Eng. Sci.* **10** 83
[50] Nowacki J P 1973 *Bull. Acad. Pol. Sci., Ser. Sci. Tech.* **21** 585
[51] Nowacki N 1974 *Arch. Mech.* **26** 3
[52] Minagawa S 1977 *Lett. Appl. Eng. Sci.* **5** 85
[53] Edelen D G B 1989 *Int. J. Eng. Sci.* **27** 653

[54] Kondo K 1952 On the geometrical and physical foundations of the theory of yielding *Proc. 2nd Japan National Congress for Applied Mechanics (Tokyo)* p 41
[55] Bilby B A, Bullough R and Smith E 1955 *Proc. R. Soc.* A **231** 263
[56] Kröner E and Seeger A 1959 *Arch. Ration. Mech. Anal.* **3** 97
[57] Kröner E 1960 *Arch. Ration. Mech. Anal.* **4** 273
[58] Hehl F W, McCrea J D, Mielke E W and Ne'eman Y 1995 *Phys. Rep.* **258** 1
[59] Nye J F 1953 *Acta Metall.* **1** 153
[60] deWit R 1970 Linear theory of static disclinations *Fundamental Aspects of Dislocation Theory* vol 1 ed J A Simmons, R deWit and R Bullough *(Natl. Bur. Stand. (U.S.), Spec. Publ. vol 317)* p 651
[61] deWit R 1973 *J. Res. Natl. Bur. Stan.* A **77** 49
[62] Mindlin R D 1965 *Int. J. Solids Struct.* **1** 265
[63] Maugin G A 1993 *Material Inhomogeneities in Elasticity* (London: Chapman and Hall)
[64] Toupin R A 1964 *Arch. Ration Mech. Anal.* **17** 85
[65] Bond W L and Andrus J 1956 *Phys. Rev.* **101** 1211
[66] Bullough R 1958 *Phys. Rev.* **110** 620
[67] Nikitenko V I and Dedukh L M 1970 *Phys. Status Solidi* a **3** 383
[68] Timpe A 1905 *Zeits. f. Math. Phys.* **52** 348
[69] Brown W F 1941 *Phys. Rev.* **60** 139
[70] Koehler J S 1941 *Phys. Rev.* **60** 397
[71] Leibfried G and Lücke K 1949 *Z. Phys.* **126** 450
[72] Kröner E 1958 Kontinuumstheorie der Versetzungen und Eigenspannungen *Erg. Angew. Math.* **5** 1
[73] Kröner E 1981 Continuum theory of defects *Physics of Defects (Les Houches, Session 35)* ed R Balian *et al* (Amsterdam: North-Holland) p 215
[74] deWit R 1973 *J. Res. Natl. Bur. Stand.* A **77** 607
[75] Read W T 1953 *Dislocations in Crystals* (New York: McGraw Hill)
[76] Teodosiu C 1982 *Elastic Models of Crystal Defects* (Berlin: Springer)
[77] Hehl F W and Kröner E 1965 *Z. Naturforsch.* a **20** 336
[78] Kröner E 1963 *Int. J. Eng. Sci.* **1** 261
[79] Kröner E and Datta B K 1966 *Z. Phys.* **196** 203
[80] Kröner E 1967 *Int. J. Solids Struct.* **3** 731